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Oscillation theorems for even order half-linear neutral differential equation with continuous deviating arguments

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Abstract. In this paper we investigate a class of even order half-linear neutral differential equation with continuous deviating arguments. By using the generalized Riccati technique and the integral averaging technique, we give some oscillatory criteria for the equation.

Key words. half-linear, neutral differential equations, continuous deviating argument.

1. Introduction

The study of oscillatory and asymptotic behavior of the solutions of even order neutral differential equations, besides its theoretical interest, is important from the viewpoint of applications. Some results concerning the oscillation and asymptotic behavior of the solutions of neutral differential equations were recently obtained by Zahariev and Bainov [1], Philos [2], Ladas and Sficas [3]. Some applicable example and basic result can be found in [4–6]. Grace discussed the Oscillation of nonlinear functional differential equation with deviating arguments and neutral nonlinear functional differential equation [7, 8]. However, very little is known for the case of half-linear neutral differential equation with continuous deviating arguments.

In this paper we consider the following even order half-linear neutral differential equations with distributed deviating arguments. By choose appropriate function $H(t;s); h(t;s)$ and $\rho(s)$, we can present a series of explicit oscillation

$$\{r(t)|[x(t)+c(t)x(t-\tau)]^{(n-1)}|^{\alpha-1}[x(t)+c(t)x(t-\tau)]^{(n-1)}\}'+ \\ +\int_a^b q(t,\xi)|x[g(t,\xi)]|^{\alpha-1}x[g(t,\xi)]d\sigma(\xi)=0\,, \tag{1}$$

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where n is an even, α and τ are positive constants.

We assume throughout this paper that the following conditions hold:

(H1) $r(t) \in C'([t_0, +\infty), R), c(t) \in C([t_0, +\infty), R), q(t, \xi) \in C([t_0, +\infty) \times [a, b], R)$;

(H2) $g(t, \xi) \in C([t_0, +\infty) \times [a, b], R), g(t, \xi) \leq t, \xi \in [a, b]$. $g(t, \xi)$ is non-decreasing with respect to t and ξ respectively, and $\lim_{t \to +\infty} \min_{\xi \in [a,b]} \{g(t, \xi)\} = +\infty$;

(H3) $\sigma(\xi) \in ([a, b], R)$ is non-decreasing, integral of Eq.(1) is a Stieltjes one.

We restrict our attention to a nontrivial solutions of Eq.(1), that is, to non-constant solutions of existing on $[T, \infty]$ for $T \geq t_0$ and satisfying $\sup_{t \geq T} |x(t)| > 0$. A nontrivial solution $x(t)$ of Eq.(1) is called oscillatory if it has arbitra-rily large zeros; otherwise it is said to be non-oscillatory. Eq.(1) is oscillatory if all of its solutions are oscillatory.

To obtain oscillatory criteria of Eq.(1), we first need the following Lemmas.

Lemma 1.1 Let $u(t)$ be a positive and n times differentiable function on R_+. If $u^{(n)}(t)$ is of constant sign and not identically zero on any ray $[t_1, +\infty)$ for $t_1 > 0$, then there exists a $t_u \geq t_1$ and an integer $l (0 \leq l \leq n)$, with $n + l$ even for $u(t)u^{(n)}(t) \geq 0$ or $n + l$ odd for $u(t)u^{(n)}(t) \leq 0$; and for $t \geq t_u$,

$$u(t)u^{(k)}(t) > 0, 0 \leq k \leq l; (-1)^{k-l} u(t)u^{(k)}(t) > 0, l \leq k \leq n.$$

Lemma 1.2 Suppose that the conditions of Lemma 1.1 are satisfied, and $u^{(n-1)}(t)$ $u^n(t) \leq 0, t \geq t_u$, then there exists a constant $\lambda \in (0, 1)$ such that for sufficiently large t, there exists a constant $M > 0$ satisfying $|u'(\lambda t)| \geq M t^{n-2} |u^{(n-1)}(t)|$.

Lemma 1.3 If X and Y are nonnegative, then $X^\lambda - \lambda X Y^{\lambda-1} + (\lambda-1) Y^\lambda \geq 0, \lambda > 1$; and $X^\lambda - \lambda X Y^{\lambda-1} - (1-\lambda) Y^\lambda \leq 0, 0 < \lambda < 1$, where the equality holds if and only if $X = Y$.

2. Main results

We can now prove the following theorems.

Theorem 2.1 Suppose that the following conditions hold:

(A_1) $0 \leq c(t) \leq 1, q(t, \xi) \geq 0$;

(A_2) $r(t) \geq 0, \displaystyle\int_{t_1}^{+\infty} \left(\frac{1}{r(s)} \right)^{\frac{1}{\alpha}} ds = +\infty.$

If $\frac{d}{dt} g(t, a)$ exists, $r(t)$ is non-decreasing and there exists a function $\varphi(t) \in C'([t_0, +\infty), (0, +\infty))$, $\varphi(t)$ is non-decreasing with respect to t, such that

$$\int_{t_1}^{+\infty} \left[\varphi(s) \int_a^b q(s, \xi) \{1 - c[g(s, \xi)]\}^\alpha d\sigma(\xi) - \right.$$
$$\left. -\lambda r(s) \varphi'(s) \left(\frac{\varphi'(s)}{M \varphi(s) [g(s, a)]^{n-2} g'(s, a)} \right)^\alpha \right] ds = +\infty \,, \tag{2}$$

where $\lambda = \frac{\alpha+1}{\alpha}$, then all solutions of Eq.(1) are oscillatory.

Proof. Suppose to the contrary that there exists a non-oscillatory solution $x(t)$ of Eq.(1). Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. From (H3) and (H2), there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(t-\tau) > 0$ and $x[g(t,\xi)] > 0$ for $t \geq t_1$, $\xi \in [a,b]$. Letting

$$z(t) = x(t) + c(t)x(t-\tau). \tag{3}$$

Then Eq.(1) can be written as

$$\left[r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\right]' + \int_a^b q(t,\xi)x[g(t,\xi)]^\alpha d\sigma(\xi) = 0.$$

From the assumption of $c(t)$ and $q(t,\xi)$, we have $z(t) \geq x(t) > 0$ and

$$\left[r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\right]' \leq 0. \tag{4}$$

We can prove $z^{(n-1)}(t) \geq 0, t \geq t_1$. In fact, suppose that $z^{(n-1)}(t) < 0, t \geq t_1$, then $r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t) < 0$. From (4) we have that $r(t)|z^{(n-1)}(t)$ $|^{\alpha-1}z^{(n-1)}(t)$ is decreasing in t, and thus

$$r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t) \leq r(t_2)|z^{(n-1)}(t_1)|^{\alpha-1}z^{(n-1)}(t_2), t \geq t_2 \geq t_1.$$

Which imply that

$$\left|z^{(n-1)}(t)\right|^{\alpha-1}z^{(n-1)}(t) \leq \frac{r(t_2)|z^{(n-1)}(t_2)|^{\alpha-1}z^{(n-1)}(t_2)}{r(t)} < 0$$

$$-z^{(n-1)}(t) = |z^{(n-1)}(t)| \geq \left(\frac{-r(t_2)|z^{(n-1)}(t_2)|^{\alpha-1}z^{(n-1)}(t_2)}{r(t)}\right)^{\frac{1}{\alpha}}$$

$$= \left(\frac{r(t_2)|z^{(n-1)}(t_2)|^{\alpha}}{r(t)}\right)^{\frac{1}{\alpha}}.$$

So we have $z^{(n-1)}(t) \leq \left(\frac{r(t_2)|z^{(n-1)}(t_2)|^{\alpha}}{r(t)}\right)^{\frac{1}{\alpha}}$. Integrating both sides of the above inequality from t_2 to t, we have

$$z^{(n-2)}(t) \leq z^{(n-2)}(t_2) - \left(r(t_2)\left|z^{(n-1)}(t_2)\right|^{\alpha}\right)^{\frac{1}{\alpha}} \int_{t_2}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds\,.$$

Letting $t \to +\infty$, from (A_2) we have $\lim_{t\to+\infty} z^{(n-2)}(t) = -\infty$, and thus $\lim_{t\to+\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Thus $z^{(n-1)}(t) \geq 0$. Furthermore, from Lemma1.1, there exists a $t_3 \geq t_2$ and an odd number l, $0 \leq l \leq n-1$, for $t \geq t_3$, we have $z^{(i)}(t) > 0, 0 \leq i \leq l; (-1)^{i-1}z^{(i)}(t) > 0, l \leq i \leq n-1$. By choosing $i = 1$, we have

$z'(t) > 0$. From (3), Eq.(1) can be written as

$$\Big[r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\Big]' + \\ + \int_a^b q(t,\xi)\{z[g(t,\xi)] - c[g(t,\xi)]x[g(t,\xi)-\tau]^{\alpha} d\sigma(\xi) = 0\,.$$

Since that $z(t) \geq x(t) > 0, z'(t) \geq 0$, we have $z[g(t,\xi)] \geq z[g(t,\xi)-\tau] \geq x[g(t,\xi)-\tau]$, and thus we have

$$\Big[r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\Big]' + \int_a^b q(t,\xi)z[g(t,\xi)]^{\alpha}\{1-c[g(t,\xi)]\}^{\alpha}d\sigma(\xi) \leq 0\,. \quad (5)$$

Since that $g(t,\xi)$ is non-decreasing in ξ, we have $g(t,a) \leq g(t,\xi)$, $t > t_0$, $\xi \in [a,b]$, thus $z[g(t,a)] \leq z[g(t,\xi)]$. Then (5) can be written as

$$\Big[r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\Big]' + z[g(t,a)]\int_a^b q(t,\xi)\{1-c[g(t,\xi)]^{\alpha}d\sigma(\xi) \leq 0\,, \quad (6)$$

where $t \geq t_1$. Letting $w(t) = \frac{\varphi(t)r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)}{z[g(t,a)]^{\alpha}}$, then $w(t) \geq 0$, for $t \geq t_1$. We have $z^{(n-1)}(t) \geq 0$, then $w(t)$ can be written as $w(t) = \frac{\varphi(t)r(t)[z^{(n-1)}(t)]^{\alpha}}{z[g(t,a)]^{\alpha}}$.

And thus

$$\begin{aligned} w'(t) = \frac{\varphi'(t)}{\varphi(t)}w(t) + \frac{\varphi(t)[r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)]'}{z[g(t,a)]^{\alpha}} \\ - \frac{\varphi(t)r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)z'[g(t,a)]g'(t,a)\alpha z[g(t,a)]^{\alpha-1}}{z[g(t,a)]^{2\alpha}}\,. \end{aligned}$$

From $\big[r(t)|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\big]' \leq 0$, $z^{(n-1)}(t) \geq 0$, and $r'(t) \geq 0$ we conclude that

$$r'(t)(z^{(n-1)}(t))^{\alpha} + \alpha r(t)(z^{(n-1)}(t))^{\alpha-1}z^{(n)}(t) \leq 0\,,$$

which implies that $z^{(n)}(t) \leq 0$. According to Lemma 1.2, we obtain $z'[g(t,a)] \geq M[g(t,a)]^{n-2}z^{n-1}(t)$ And thus

$$\begin{aligned} w'(t) &\leq \frac{\varphi'(t)}{\varphi(t)}w(t) - \varphi(t)\int_a^b q(t,\xi)\{1-c[g(t,\xi)]\}^{\alpha}d\sigma(\xi) \\ &\quad - \frac{\alpha\varphi(t)r(t)[z^{(n-1)}(t)]^{\alpha+1}M[g(t,a)]^{n-2}g'(t,a)}{z[g(t,a)]^{\alpha+1}} \\ &= -\varphi(t)\int_a^b q(t,\xi)\{1-c[g(t,\xi)]\}^{\alpha}d\sigma(\xi) \\ &\quad + \frac{\varphi'(t)}{\varphi(t)}w(t) - \alpha M[g(t,a)]^{n-2}g'(t,a)[r(t)\varphi(t)]^{\frac{1}{\alpha}}w(t)^{\frac{\alpha+1}{\alpha}}\,. \end{aligned} \quad (7)$$

Taking

$$X = \frac{\left(\alpha M[g(t,a)]^{n-2}g'(t,a)\right)^{\frac{\alpha}{\alpha+1}} w(t)}{[r(t)\varphi(t)]^{\frac{1}{\alpha+1}}}, \lambda = \frac{\alpha+1}{\alpha},$$
$$Y = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \left[\frac{\varphi'(t)}{\varphi(t)}(r(t)\varphi(t))^{\frac{1}{\alpha+1}}(\alpha M[g(t,a)]^{n-2}g'(t,a))^{\frac{-\alpha}{\alpha+1}}\right]^{\alpha}.$$

According to Lemma 1.3, we obtain

$$\frac{\varphi'(t)}{\varphi(t)}w(t) - \alpha M[g(t,a)]^{n-2}g'(t,a)[r(t)\varphi(t)]^{-\frac{1}{\alpha}}w(t)^{\frac{\alpha+1}{\alpha}}$$
$$\leq \lambda r(t)\varphi(t)\left(\frac{\varphi'(t)}{\varphi(t)}\right)^{\alpha+1}\left(M[g(t,a)]^{n-2}g'(t,a)\right)^{-\alpha},$$

thus

$$w'(t) \leq -\varphi(t)\Bigg[\int_a^b q(t,\xi)\{1-c[g(t,\xi)]\}^{\alpha}d\sigma(\xi) - \frac{\lambda r(t)\varphi'(t)}{\varphi(t)}\left(\frac{\varphi'(t)}{M[g(t,a)]^{n-2}g'(t,a)\varphi(t)}\right)^{\alpha}\Bigg]. \tag{8}$$

Integrating both sides from t_1 to t, we have

$$w(t) \leq w(t_1) - \int_{t_1}^{t}[\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\sigma(\xi) - \lambda r(s)\varphi'(s)\left(\frac{\varphi'(s)}{M[g(s,a)]^{n-2}g'(s,a)\varphi(s)}\right)^{\alpha}]ds.$$

Letting $t \to +\infty$, from (2), we have $\lim\limits_{t\to+\infty} w(t) = -\infty$, which leads to a contradiction with $w(t) > 0$. This completes the proof of Theorem 2.1.

Theorem 2.2 Suppose that conditions (A_1) and (A_2) hold and $r(t)$ is non-decreasing. If there exists a $\frac{d}{dt}g(t,a)$ and there exist function $\varphi(t), \rho(s) \in C'([t_0,+\infty),(0,+\infty))$, $\varphi(t)$ is non-decreasing with respect to t. Letting function $H(t,s), h(t,s) \in C'(D,R)$, in which $D = \{(t,s)|t \geq s \geq t_0\}$, such that

(H4) $H(t,t) = 0, t \geq t_0; H(t,s) > 0, t > s \geq t_0$;

(H5) $H'(t,s) \geq 0, H'_s(t,s) \leq 0$;

(H6) $-\frac{\partial[H(t,s)\rho(s)]}{\partial s} - H(t,s)\rho(s)\frac{\varphi'(s)}{\varphi(s)} = h(t,s)$,

If

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\sigma(\xi) - \frac{\beta r(s)\varphi(s)|h(t,s)|^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}}]ds = +\infty, \tag{9}$$

where $\beta=\left(\frac{1}{\alpha+1}\right)^{\alpha+1}$. Then all solutions of Eq.(1) are oscillatory.

3. The example

The following example illustrates our theory.
Example 2.1 Consider the 4-order equation

$$\begin{aligned}&\{|[x(t)+(1-e^{-t/\alpha})x(t-\tau)]^{(3)}|^{\alpha-1}[x(t)+(1-e^{-t/\alpha})x(t-\tau)]^{(3)}\}'\\&+\int_{-1}^{0}e^{2t+2\xi}|x(t,\xi)|^{\alpha-1}x(t+\xi)d\xi=0\,.\end{aligned}\tag{10}$$

Choosing $\varphi(t)=t$, then the conditions of (A_1), (A_2) hold, and we have

$$\begin{aligned}&\int_{t_1}^{+\infty}\left[s\int_{-1}^{0}e^{2s+2\xi}\{1-1+e^{-(s+\xi)/\alpha}\}^{\alpha}d\xi-\lambda\frac{1}{(Ms(s-1)^2)^{\alpha}}\right]ds\\=&\int_{t_1}^{+\infty}\left[s\int_{-1}^{0}e^{2s+2\xi}e^{-(s+\xi)}d\xi-\lambda\left(\frac{1}{M}\right)^{\alpha}\frac{1}{(s(s-1)^2)^{\alpha}}\right]ds\\=&\int_{t_1}^{+\infty}se^{s}ds-\int_{t_1}^{+\infty}se^{s-1}ds-\frac{\lambda}{M^{\alpha}}\int_{t_1}^{+\infty}\frac{1}{(s(s-1)^2)^{\alpha}}ds=+\infty\,.\end{aligned}$$

Therefore, all solution of equation (10) are oscillatory by Theorem 2.1.
Example 2.2 Consider the high-order equation for $n=m+2, m$ is an even.

$$\begin{aligned}&\left\{\left|[x(t)+(1-\frac{1}{t})x(t-\tau)]^{(m+1)}\right|^{\alpha-1}[x(t)+(1-\frac{1}{t})x(t-\tau)]^{(m+1)}\right\}'\\&+\int_{\frac{1}{2}}^{1}(t^2\xi)^{\alpha}|x(t\xi)|^{\alpha-1}x(t\xi)d\xi=0\,.\end{aligned}\tag{11}$$

The conditions of $(A_1),(A_2)$ hold, taking $\varphi(t)=t^2,\rho(t)=\frac{1}{t^2}$, $H(t,s)=(t-s)^2$, for $t\geq s\geq t_0$. Then the conditions of (H4), (H5) in Theorem 2.2 are satisfied, and we have
$h(t,s)=2(t-s)/s^2$. Thus we conclude that

$$\begin{aligned}&\limsup_{t\to+\infty}\frac{1}{(t-t_0)}\int_{t_0}^{t}(t-s)^2\int_{\frac{1}{2}}^{1}(s^2\xi)^{\alpha}\left(\frac{1}{s\xi}\right)^{\alpha}d\xi ds\\=&\limsup_{t\to+\infty}\frac{1}{2(t-t_0)}\int_{t_0}^{t}s^{\alpha}(t-s)^2ds\\=&\limsup_{t\to+\infty}\frac{1}{2(t-t_0)}\left\{t^{\alpha+3}\left(\frac{1}{\alpha+1}+\frac{1}{\alpha+3}-\frac{2}{\alpha+2}\right)\right.\\&\left.-\left(\frac{t^2t_0^{\alpha+1}}{\alpha+1}+\frac{t_0^{\alpha+3}}{\alpha+3}+\frac{2tt_0^{\alpha+2}}{\alpha+2}\right)\right\}\end{aligned}$$

$$= +\infty\,.$$

On the other hand

$$\int_{t_0}^{t} \frac{\beta s^2 \left[\frac{2(t-s)}{s^2}\right]^{\alpha+1}}{\left(M(t-s)^2 \frac{1}{s^2}\left(\frac{s}{2}\right)^m \frac{1}{2}\right)^{\alpha}} ds \le \frac{\beta 2^{(m+2)\alpha+1}}{M^{\alpha}} \times \frac{(t-t_0)^{2-\alpha}}{\alpha-2} \times \frac{\left(t^{1-m\alpha} - t_0^{1-m\alpha}\right)}{1-m\alpha}\,.$$

When $\alpha > 2$, we have that $\lim\sup_{n\to+\infty} \int_{t_0}^{t} \frac{\beta s^2 \left[\frac{2(t-s)}{s^2}\right]^{\alpha+1}}{\left(M(t-s)^2 \frac{1}{s^2}\left(\frac{s}{2}\right)^m \frac{1}{2}\right)^{\alpha}} ds = 0$. Therefore

$$\lim\sup_{t\to+\infty} \frac{1}{(t-t_0)} \int_{t_0}^{t} \left[(t-s)^2 \int_{\frac{1}{2}}^{1} (s^2\xi)^{\alpha} \left(\frac{1}{s\xi}\right)^{\alpha} d\xi - \frac{\beta s^2 \left[\frac{2(t-s)}{s^2}\right]^{\alpha+1}}{\left(M(t-s)^2 \frac{1}{s^2}\left(\frac{s}{2}\right)^m \frac{1}{2}\right)^{\alpha}}\right] ds$$
$$= +\infty\,.$$

So that (9) is satisfied. Consequently, all solutions of equation (11) are oscillatory by Theorem 2.2.

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